

CALABI-YAU EXTENSION AND LOCALIZATION OF KOSZUL REGULAR ALGEBRAS

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ABSTRACT. Let A be a Koszul Artin-Schelter regular algebra, θ be an algebra automorphism of A and σ be an algebra homomorphism from A to $M_{2 \times 2}(A)$. We compute the Nakayama automorphisms of a skew polynomial extension $A[t; \theta]$ and of a trimmed double Ore extension $A_P[y_1, y_2; \sigma]$ (introduced in [ZZ08]) respectively. This leads to a characterization of the Calabi-Yau property of $A_P[y_1, y_2; \sigma]$, the skew Laurent extension $A[t^{\pm 1}; \theta]$ and $A[y_1^{\pm 1}, y_2^{\pm 1}; \sigma]$ with σ a diagonal type.

INTRODUCTION

Let A be a Koszul Artin-Schelter regular algebra with Nakayama automorphism ν in the sense of [BZ08]. In [HVZ13], the authors proved that the skew polynomial extension $A[t; \nu]$ is Calabi-Yau. Similar results have been proved in [BOZZ13, GK13, GYZ14, RRZ13, LWW12]. In this note, we first study the Calabi-Yau property of a certain double Ore extension $A_P[y_1, y_2; \sigma]$ of A ; the general notion of a double Ore extension was introduced by Zhang and Zhang in [ZZ08]. We then apply the method to study the Calabi-Yau property of a skew Laurent extension $A[t^{\pm 1}; \theta]$, where $\theta \in \text{Aut}(A)$, and $A[y_1^{\pm 1}, y_2^{\pm 1}; \sigma]$ with σ a diagonal type. Our first main result reads as follows.

Theorem 1. (Theorem 2.12) *A so-called trimmed double Ore extension $A_P[y_1, y_2; \sigma]$ of a Koszul Artin-Schelter regular algebra A is Calabi-Yau if and only if $\det_r \sigma = \nu$ and a homological determinant type condition is satisfied. Here, $\det_r \sigma$ is an algebra automorphism induced by σ .*

It follows from Farinati's result on the Van den Bergh duality [F05, Theorem 6] that the Calabi-Yau property is preserved under noncommutative localizations. Here, we characterize the Calabi-Yau property of the localization of both the skew polynomial extension with respect to the Ore set $\{t^i, i \in \mathbb{N}\}$, which is often called the skew Laurent extension, and the iterated skew polynomial extension. The second main result is the following:

Theorem 2. (Theorem 3.2 and Theorem 3.5) *Let A be a Koszul Artin-Schelter regular algebra with Nakayama automorphism ν .*

(1) *The skew Laurent extension $A[t^{\pm 1}; \theta]$ of A is Calabi-Yau if and only if there exists an integer n such that $\theta^n = \nu$ and the homological determinant $\text{hdet}(\theta)$ of θ*

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equals 1.

(2) Given two automorphisms τ and ξ of A , let, $Q = A[y_1^{\pm 1}, y_2^{\pm 1}; \sigma]$, where $\sigma = \text{diag}(\tau, \xi)$ is a map from A to $M_{2 \times 2}(A)$. Then, Q is Calabi-Yau if and only if there exists two integers m, n such that $\tau^m \xi^n = \nu$ and $\text{hdet}(\tau) = \text{hdet}(\xi) = 1$.

The proofs of the main results are based on the computation of the Nakayama automorphisms of algebras $A[t; \theta]$, $A_P[y_1, y_2; \sigma]$, and their localizations. It turns out that there exists a strong relation between the Nakayama automorphisms of these extensions and the homological determinants (see the paragraph above Prop. 1.5 for the notion). In [CLM11], the relation between double Ore extensions and iterated Ore extensions was discussed. Further, the Nakayama automorphisms of right coideal subalgebras of quantized enveloping algebras were explicitly computed in [LW14, Theorem 0.1]. These coideal subalgebras are in fact iterated Ore extensions in a special case. So it would be interesting to study the Calabi-Yau property of the double Ore extensions (and the localizations) of iterated skew polynomial extensions in general. In Theorem 3.6, the Calabi-Yau property of a class of iterated skew polynomial extensions and the Calabi-Yau property of their localizations are studied. Necessary and sufficient conditions for those algebras to be Calabi-Yau are determined.

The paper is organized as follows. In Section 1, we recall some preliminary results including the relation of the Nakayama automorphism for Koszul Artin-Schelter regular algebras and their Yoneda Ext algebras, and some basic materials on double Ore extensions. In Section 2, we first prove that the Koszulity is preserved under trimmed double Ore extensions (Proposition 2.1) and then study the Nakayama automorphism and the Calabi-Yau property of trimmed double Ore extensions. In Section 3 we present similar results for the skew Laurent extension $A[t^{\pm 1}; \theta]$, iterated skew Laurent extensions.

Throughout, \mathbb{k} is a field and all algebras are \mathbb{k} -algebras; unadorned \otimes means $\otimes_{\mathbb{k}}$, Hom means $\text{Hom}_{\mathbb{k}}$ and $*$ always denotes the dual over \mathbb{k} .

1. PRELIMINARIES

An \mathbb{N} -graded algebra $A = \bigoplus_{i \geq 0} A_i$ is called connected if $A_0 = \mathbb{k}$. By a graded algebra we mean a locally finite graded algebra generated in degree 1. A module means a left (graded) module. Shifting of a graded module is denoted by $(\)$. For a module M over A , ${}^\varphi M$ stands for a twisted module by an algebra automorphism φ , where the action is defined by $a \cdot m := \varphi(a)m$. Similarly, M^φ and ${}^1 M^\varphi$ denote the twisted right module and the twisted bimodule respectively.

Let V be a finite-dimensional vector space, and $T_{\mathbb{k}}(V)$ be the tensor algebra with the usual grading. A connected graded algebra $A = T_{\mathbb{k}}(V)/\langle R \rangle$ is called a quadratic algebra if R is a subspace of $V^{\otimes 2}$. The homogeneous dual of A is then defined as $A^! = T_{\mathbb{k}}(V^*)/\langle R^\perp \rangle$, where R^\perp is the orthogonal complement of R in $(V^*)^{\otimes 2}$.

Definition 1.1. A quadratic algebra A is called Koszul if the trivial A -module ${}_A \mathbb{k}$ admits a projective resolution

$$\cdots \longrightarrow P_n \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow {}_A \mathbb{k} \longrightarrow 0$$

such that P_n is generated in degree n for all $n \geq 0$.

For more details about Koszul algebras and the Koszul duality, we refer to Chapter 2 of Polishchuk and Positselski's text book [PP05]. Now, recall the definitions of an Artin-Schelter regular algebra, a Nakayama automorphism and a Calabi-Yau algebra.

Definition 1.2. *A connected graded algebra A is called Artin-Schelter (AS, for short) Gorenstein of dimension d with parameter l for some integers d and l , if*

- (i) $\text{inj. dim}(A_A) = \text{inj. dim}(A_A) = d$; and
- (ii) $\text{Ext}_A^i(\mathbb{k}, A) \cong \text{Ext}_{A^{op}}^i(\mathbb{k}, A) \cong \begin{cases} 0, & i \neq d, \\ \mathbb{k}(l), & i = d. \end{cases}$

If, furthermore, A has a finite global dimension, then A is called AS-regular.

Definition 1.3. [G06, BZ08] *A graded algebra A is called twisted Calabi-Yau of dimension d if*

- (i) *A is homologically smooth, i.e., A , as an A^e -module, has a finitely generated projective resolution of finite length.*
- (ii) $\text{Ext}_{A^e}^i(A, A^e) \cong \begin{cases} 0, & i \neq d \\ A^\nu(l), & i = d \end{cases}$ *as A^e -modules for some automorphism ν of A and integers d, l .*

The automorphism ν is called the Nakayama automorphism of A . If, in addition, A^ν is isomorphic to A as A^e -modules, or equivalently, ν is inner, then A is called Calabi-Yau of dimension d . Ungraded Calabi-Yau algebras are defined similarly but without degree shift.

Let E be a Frobenius algebra. By definition, there is an isomorphism $\varphi : E \cong E^*$ of right E -modules. This is equivalent to the existence of a nondegenerate bilinear form, often called Frobenius pair, $\langle -, - \rangle : E \times E \rightarrow \mathbb{k}$ such that $\langle ab, c \rangle = \langle a, bc \rangle$ for all $a, b, c \in E$ (where the bilinear form is defined by $\langle a, b \rangle := \varphi(b)(a)$). By the nondegeneracy of the bilinear form, there exists an automorphism μ , unique up to an inner automorphism, such that

$$(1.1) \quad \langle a, b \rangle = \langle \mu(b), a \rangle$$

for all $a, b \in E$. Thus, φ becomes an isomorphism of E -bimodules ${}^\mu E \cong E^*$. The automorphism μ is usually called the Nakayama automorphism of E . For more detail, see [Sm96].

Now, there are two notions of Nakayama automorphisms: one for twisted Calabi-Yau algebras and one for Frobenius algebras. We use ν for the former and μ for the latter if there is no confusion. In fact, the notion of a Nakayama automorphism in [BZ08] can be defined for algebras with finite injective dimension, and it coincides with the classical Nakayama automorphism of a Frobenius algebra. But in this paper, we focus ourselves on twisted Calabi-Yau algebras (or equivalently, AS-regular algebras in the graded case). It is well known that a connected graded algebra A is AS-regular if and only if its Yoneda Ext algebra is Frobenius [LPWZ08, Corollary D]. In this case, the two notions of Nakayama automorphisms will coincide in the sense of the Koszul duality, see Proposition 1.4. For this end, we need the following preparation.

Let $A = T_{\mathbb{k}}(V)/\langle R \rangle$ be a Koszul algebra. Then its Yoneda Ext algebra $E(A)$ is isomorphic to $T_{\mathbb{k}}(V^*)/\langle R^\perp \rangle$, see [Sm96]. Let θ be a graded automorphism of A , and define $\theta^* : V^* \rightarrow V^*$ by $\theta(f)(x) = f(\theta(x))$ for each $f \in V^*$ and $x \in V$. It is easy to see that θ^* induces a graded automorphism of $E(A)$ because θ is assumed to preserve the relation R . We still use the notation θ^* for this algebra automorphism. Suppose that $\{e_1, e_2, \dots, e_n\}$ is a \mathbb{k} -basis of V and $\{e_1^*, e_2^*, \dots, e_n^*\}$ the corresponding dual basis of V^* . If $\theta(e_i) = \sum_j c_{ij} e_j$ for $c_{ij} \in \mathbb{k}$ ($1 \leq i, j \leq n$), then we have:

$$(1.2) \quad \theta^*(e_i^*) = \sum_j c_{ji} e_j^*.$$

Proposition 1.4. [VdB97, Theorem 9.2] *Let A be a Koszul AS-regular algebra of dimension d . Then, the Nakayama automorphism ν of A is equal to $\epsilon^{d+1}\mu^*$, where μ is the Nakayama automorphism of the Frobenius algebra $A^!$ and ϵ is the automorphism of A defined by $a \mapsto (-1)^{\deg a}a$, for any homogeneous element $a \in A$.*

In order to determine the Nakayama automorphisms of the algebras considered in this paper, we need the notion of the homological determinant. Roughly speaking, for an AS-Gorenstein algebra A , the homological determinant, hdet , is a homomorphism from the graded automorphism group $\text{GrAut}(A)$ of A to the multiplicative group $\mathbb{k} \setminus \{0\}$ generalizing the usual determinant of a matrix [JZ00]. For the precise definition and its application, we refer to [JZ00, RRZ13]. Here, we just need the following characterization of the homological determinant of an automorphism of a Koszul algebra.

Proposition 1.5. [WZ11, Proposition 1.11] *Let A be a Koszul AS-regular algebra of global dimension d . Suppose that σ is a graded automorphism of A and σ^* is its extension to a dual graded automorphism for the dual algebra $A^!$. Then, we have $\sigma^*(u) = (\text{hdet } \sigma)u$ for any $u \in \text{Ext}_A^d(\mathbb{k}, \mathbb{k})$.*

Next, we recall the definition and some basic properties of a double Ore extension.

Definition 1.6. [ZZ08, ZZ09] *Let A be a subalgebra of a \mathbb{k} -algebra B . Then:*

(1). *B is called a right double Ore extension of A if:*

- (i) *B is generated by A and two new variables y_1 and y_2 ;*
- (ii) *y_1 and y_2 satisfy the relation*

$$y_2 y_1 = p y_1 y_2 + q y_1^2 + \tau_1 y_1 + \tau_2 y_2 + \tau_0$$

for some $p, q \in \mathbb{k}$ and $\tau_1, \tau_2, \tau_0 \in A$;

- (iii) *B is a free left A -module with basis $\{y_1^i y_2^j; i, j \geq 0\}$;*
- (iv) *$y_1 A + y_2 A + A \subseteq A y_1 + A y_2 + A$.*

(2). *B is called a left double Ore extension of A if:*

- (i) *B is generated by A and two new variables y_1 and y_2 ;*
- (ii) *y_1 and y_2 satisfy the relation*

$$y_1 y_2 = p' y_2 y_1 + q' y_1^2 + \tau'_1 y_1 + \tau'_2 y_2 + \tau'_0$$

for some $p', q' \in \mathbb{k}$ and $\tau'_1, \tau'_2, \tau'_0 \in A$;

- (iii) B is a free right A -module with basis $\{y_2^i y_1^j; i, j \geq 0\}$;
- (iv) $Ay_1 + Ay_2 \subseteq y_1 A + y_2 A + A$.

(3). B is called a double Ore extension of A if it is a left and a right double Ore extension of A with the same generators $\{y_1, y_2\}$.

Condition (1).(iv) in the above definition is equivalent to the existence of two maps:

$$\sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix} : A \rightarrow M_{2 \times 2}(A) \quad \text{and} \quad \delta = \begin{pmatrix} \delta_1 \\ \delta_2 \end{pmatrix} : A \rightarrow M_{2 \times 1}(A)$$

subject to

$$(1.3) \quad \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} a = \sigma(a) \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \delta(a)$$

for all $a \in A$. In the case B is a right double Ore extension of A , we will write $B = A_P[y_1, y_2; \sigma, \delta, \tau]$, where $P = \{p, q\} \subset \mathbb{k}$, $\tau = \{\tau_0, \tau_1, \tau_2\} \subset A$, and σ, δ as above. Similar to the Ore extension, σ is a homomorphism of algebras and δ is a σ -derivation, that is, δ is \mathbb{k} -linear and satisfies $\delta(ab) = \sigma(a)\delta(b) + \delta(a)b$, for all $a, b \in A$. The double Ore extensions that we consider mainly in this work are the following

Definition 1.7. A right double extension $A_P[y_1, y_2; \sigma, \delta, \tau]$ is called a trimmed right double Ore extension, denoted it by $A_P[y_1, y_2; \sigma]$, if δ is a zero map and $\tau = \{0\}$.

Dually, Condition (2).(iv) in the above definition is equivalent to the existence of two maps

$$\phi = \begin{pmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{pmatrix} : A \rightarrow M_{2 \times 2}(A) \quad \text{and} \quad \delta' = \begin{pmatrix} \delta'_1 & \delta'_2 \end{pmatrix} : A \rightarrow M_{1 \times 2}(A)$$

satisfying

$$(1.4) \quad a \begin{pmatrix} y_1 & y_2 \end{pmatrix} = \begin{pmatrix} y_1 & y_2 \end{pmatrix} \phi(a) + \delta'(a)$$

for all $a \in A$. For a double Ore extension, the relation of equations (1.3) and (1.4) will be explained in the following.

Definition 1.8. [ZZ08] Let $\sigma : A \rightarrow M_{2 \times 2}(A)$ be an algebra homomorphism. We say that σ is invertible if there is an algebra homomorphism $\phi = \begin{pmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{pmatrix} : A \rightarrow M_{2 \times 2}(A)$ satisfies the following conditions:

$$\sum_{k=1}^2 \phi_{jk}(\sigma_{ik}(r)) = \begin{cases} r, & i = j \\ 0, & i \neq j \end{cases} \quad \text{and} \quad \sum_{k=1}^2 \sigma_{kj}(\phi_{ki}(r)) = \begin{cases} r, & i = j \\ 0, & i \neq j \end{cases}$$

for all $r \in A$. The map ϕ is called the inverse of σ .

Lemma 1.9. [ZZ08, Lemma 1.9, Proposition 2.1] Let $B = A_P[y_1, y_2; \sigma, \delta, \tau]$ be a right double Ore extension of A .

- (1) If B is a double Ore extension, then σ is invertible.
- (2) Suppose that both A and B are connected graded algebras. If $p \neq 0$ and σ is invertible, then B is a double Ore extension.

In order to study the regularity of double Ore extensions, Zhang and Zhang introduced an invariant of σ , called the (right) determinant of σ , which is similar to the quantum determinant of the 2×2 -matrix. As we will see, it will play an important role in the description of the Nakayama automorphism of a double Ore extension.

Let $B = A_P[y_1, y_2; \sigma, \delta, \tau]$ be a right double Ore extension of A . The right determinant of σ is defined to be the map:

$$(1.5) \quad \det_r \sigma : a \mapsto -q\sigma_{12}(\sigma_{11}(a)) + \sigma_{22}(\sigma_{11}(a)) - p\sigma_{12}(\sigma_{21}(a))$$

for each $a \in A$. If σ is invertible with the inverse ϕ , then the left determinant of ϕ is defined by:

$$\det_l \phi := -q\phi_{11} \circ \phi_{21} + \phi_{11} \circ \phi_{22} - p\phi_{12} \circ \phi_{21}.$$

Remark that when $q = 0$ the above expression of $\det_l \phi$ coincides with the one in [ZZ08] after E2.1.6. The following properties of the determinant of σ were given in [ZZ08].

Proposition 1.10. [ZZ08, Proposition 2.1] *Let $B = A_P[y_1, y_2; \sigma, \delta, \tau]$ be a right double Ore extension of A , and σ be invertible with inverse ϕ . Then,*

- (1) $\det_r \sigma$ is an algebra endomorphism of A ;
- (2) if $p \neq 0$, then

$$\begin{aligned} \det_r \sigma &= \frac{q}{p}\sigma_{11} \circ \sigma_{12} + \sigma_{11} \circ \sigma_{22} - \frac{1}{p}\sigma_{21} \circ \sigma_{12}, \\ \det_l \phi &= \frac{q}{p}\phi_{21} \circ \phi_{11} + \phi_{22} \circ \phi_{11} - \frac{1}{p}\phi_{21} \circ \phi_{12}; \end{aligned}$$

- (3) $\det_r \sigma$ is invertible with inverse $\det_l \phi$.

Double Ore extensions can be used to construct AS-regular algebras from lower dimensional ones due to the following result which we will use later.

Lemma 1.11. [ZZ08, Theorem 0.2] *Let A be an AS-regular algebra. If B is a connected graded double Ore extension of A , then B is AS-regular and $\text{gldim } B = \text{gldim } A + 2$.*

2. DOUBLE ORE EXTENSIONS

In this section, we mainly study the Calabi-Yau property of a trimmed double Ore extension. First of all, we discuss the Koszul property of such extensions.

Proposition 2.1. *Let A be a Koszul algebra and $B = A_P[y_1, y_2; \sigma]$ be a trimmed right double Ore extension of A . Then, B is a Koszul algebra.*

Proof. Suppose that M is a B - A -bimodule and φ is an automorphism of A . Recall that ${}^1M^\varphi$ is the twisted bimodule on the \mathbb{k} -space M with

$$b \cdot m \cdot a = bm\varphi(a)$$

for all $m \in M, b \in B$ and $a \in A$. On the space $M \oplus M$, there is another right A -module structure defined by using σ as follows:

$$(2.1) \quad (s, t) \circ a = (s, t) \begin{pmatrix} \sigma_{11}(a) & \sigma_{12}(a) \\ \sigma_{21}(a) & \sigma_{22}(a) \end{pmatrix} = (s\sigma_{11}(a) + t\sigma_{21}(a), s\sigma_{12}(a) + t\sigma_{22}(a))$$

for all $s, t \in M$ and $a \in A$. Since σ is an algebra homomorphism, $M \oplus M$ is also a B - A -bimodule with the original left B -action and the right A -action defined by (2.1). Denote by $(M \oplus M)^\sigma$ this B - A -bimodule. By [ZZ08, Theorem 2.2], there is an exact sequence of B - A -bimodules

$$(2.2) \quad 0 \rightarrow B^{\det_r \sigma} \xrightarrow{g} (B \oplus B)^\sigma \xrightarrow{f} B \xrightarrow{\varepsilon} A \rightarrow 0,$$

where, f maps (s, t) to $sy_1 + ty_2$, g sends r to $(r(qy_1 - y_2), rpy_1)$ and the last term A is identified with $B/(y_1, y_2)$. Moreover, (2.2) is a linear resolution of A as a left B -module if we assume that both y_1 and y_2 are of degree 1.

Now by assumption, ${}_A \mathbb{k}$ admits a projective resolution:

$$(2.3) \quad \cdots \rightarrow P_n \rightarrow P_{n-1} \rightarrow \cdots \rightarrow P_1 \rightarrow P_0 \rightarrow {}_A \mathbb{k} \rightarrow 0$$

with P_n generated in degree n for each $n \geq 0$. We consider the third quadrant bicomplex:

$$\begin{array}{ccccccc} & & 0 & & 0 & & 0 \\ & & \uparrow & & \uparrow & & \uparrow \\ \cdots & \longrightarrow & B \otimes_A P_2 & \longrightarrow & B \otimes_A P_1 & \longrightarrow & B \otimes_A P_0 \longrightarrow 0 \\ & & \uparrow & & \uparrow & & \uparrow \\ \cdots & \longrightarrow & (B \oplus B)^\sigma \otimes_A P_2 & \longrightarrow & (B \oplus B)^\sigma \otimes_A P_1 & \longrightarrow & (B \oplus B)^\sigma \otimes_A P_0 \longrightarrow 0 \\ & & \uparrow & & \uparrow & & \uparrow \\ \cdots & \longrightarrow & B^{\det_r \sigma} \otimes_A P_2 & \longrightarrow & B^{\det_r \sigma} \otimes_A P_1 & \longrightarrow & B^{\det_r \sigma} \otimes_A P_0 \longrightarrow 0 \\ & & \uparrow & & \uparrow & & \uparrow \\ & & 0 & & 0 & & 0 \end{array}$$

Since each term in the sequence (2.2) is projective as a right A -module, all the rows of the bicomplex are exact except at the (-1) -th column. Thus, the homology along the rows yields a single nonzero column, that is,

$$(2.4) \quad \cdots \rightarrow 0 \rightarrow B^{\det_r \sigma} \otimes_A \mathbb{k} \rightarrow (B \oplus B)^\sigma \otimes_A \mathbb{k} \rightarrow B \otimes_A \mathbb{k} \rightarrow 0.$$

Moreover, the sequence (2.2) is a split exact sequence. Therefore, the homology of (2.4) is ${}_B A \otimes_A \mathbb{k} = {}_B \mathbb{k}$. Namely, the total complex of the bicomplex is a projective resolution of the B -module ${}_B \mathbb{k}$ and this completes the proof. \square

Suppose that $A = T_{\mathbb{k}}(V)/\langle R \rangle$ is a Koszul algebra. Let $\sigma : A \rightarrow M_{2 \times 2}(A)$ be an algebra homomorphism. Then,

$$\begin{pmatrix} \sigma_{11}^* & \sigma_{12}^* \\ \sigma_{21}^* & \sigma_{22}^* \end{pmatrix} : V^* \rightarrow M_{2 \times 2}(V^*)$$

defines a \mathbb{k} -linear map, denoted by σ^* . Extend σ^* to an algebra homomorphism $\sigma^* : T_{\mathbb{k}}(V^*) \rightarrow M_{2 \times 2}(T_{\mathbb{k}}(V^*))$ by letting:

$$\sigma^*(xy) := \sigma^*(x)\sigma^*(y)$$

for each $x, y \in T_{\mathbb{k}}(V^*)$. Furthermore, it is easy to see that σ^* induces an algebra homomorphism from $A^!$ to $M_{2 \times 2}(A^!)$ because $\sigma : A \rightarrow M_{2 \times 2}(A)$ is assumed to be an algebra homomorphism. We still use the notation σ^* for this algebra homomorphism if no confusion occurs. Moreover, similar to the relation between the automorphism group of a Koszul algebra and the one of its Koszul dual, we have the following easy property.

Lemma 2.2. *σ is invertible with an inverse ϕ if and only if σ^* is invertible with the inverse ϕ^* .*

In the rest of this section, $A = T_{\mathbb{k}}(V)/\langle R \rangle$ is a Koszul AS-regular algebra of global dimension d with Nakayama automorphism ν , and $B = A_P[y_1, y_2; \sigma]$ is a trimmed double Ore extension of A . Let $\{e_1, e_2, \dots, e_n\}$ is a \mathbb{k} -basis of V and $\{e_1^*, e_2^*, \dots, e_n^*\}$ is the corresponding dual basis of V^* . Since $A^!$ is a Frobenius algebra, then by [Sm96, Lemma 3.2] the nondegenerate bilinear form on $A^!$ is given by

$$(2.5) \quad \langle a, b \rangle = \text{the coefficient of the component of } ab \text{ in } A_d^!.$$

Let δ be a base element of the 1-dimensional space $A_d^!$. We can pick a basis $\{\eta_1, \eta_2, \dots, \eta_n\}$ of $A_{d-1}^!$ such that $e_i^* \eta_j = \delta_{ij} \delta$. Then $\eta_i e_j^* = \lambda_{ij} \delta$ for some $\lambda_{ij} \in k$. Equivalently,

$$\langle e_i^*, \eta_j \rangle = \delta_{ij}, \quad \langle \eta_i, e_j^* \rangle = \lambda_{ij}$$

for $i, j = 1, 2, \dots, n$. It follows from (1.1) that the Nakayama automorphism $\mu_{A^!}$ of $A^!$ is given by:

$$(2.6) \quad \mu_{A^!}(e_i^*) = \sum \lambda_{ji} e_j^*.$$

Assume that $B = T_{\mathbb{k}}(V \oplus \mathbb{k}y_1 \oplus \mathbb{k}y_2)/\langle R_B \rangle$. The generating relations in B are of following three types:

- (1) the relations defining A ;
- (2) $y_2 y_1 - p y_1 y_2 - q y_1^2$;
- (3) $\{y_j e_i - \sigma_{j1}(e_i) y_1 - \sigma_{j2}(e_i) y_2; j = 1, 2, i = 1, \dots, n\}$.

By Definition 1.6 and Definition 1.8, the relation (3) in the above is equivalent to

$$(3') \quad \{e_i y_j - y_1 \phi_{1j}(e_i) - y_2 \phi_{2j}(e_i); j = 1, 2, i = 1, \dots, n\}.$$

For convenience, we list the following well-known property of the algebra $C := \mathbb{k}\langle y_1, y_2 \rangle / (y_2 y_1 - p y_1 y_2 - q y_1^2) (p \neq 0)$.

Proposition 2.3. *The algebra C is Koszul AS-regular of dimension 2. Its Koszul dual $C^!$ is $\mathbb{k}\langle y_1^*, y_2^* \rangle / \langle (y_1^*)^2 + q y_2^* y_1^*, y_1^* y_2^* + p y_2^* y_1^*, (y_2^*)^2 \rangle$.*

Obviously, $\{e_1^*, e_2^*, \dots, e_n^*, y_1^*, y_2^*\}$ is a basis for $B_1^!$. Now we can characterize the dual algebra of B .

Lemma 2.4. *The generating relations for $B^!$ consist of*

- (1) the relations for $A^!$;
- (2) the relations for $C^!$;
- (3) $\{y_j^* e_i^* + \phi_{j1}^*(e_i^*) y_1^* + \phi_{j2}^*(e_i^*) y_2^*; j = 1, 2, i = 1, \dots, n\}$, where ϕ is the inverse of σ .

Proof. According to the description of the generating relations of B , all of the relations listed above in (1), (2) and (3) belong to $(R_B)^\perp$.

On the other hand, it suffices to show that every element $f = \sum_i k_i e_i^* y_1^* + l_i e_i^* y_2^* + m_i y_1^* e_i^* + n_i y_2^* e_i^* \in (R_B)^\perp$ can be written as

$$f = \sum a_i (y_1^* e_i^* + \phi_{11}^*(e_i^*) y_1^* + \phi_{12}^*(e_i^*) y_2^*) + b_i (y_2^* e_i^* + \phi_{21}^*(e_i^*) y_1^* + \phi_{22}^*(e_i^*) y_2^*),$$

for $a_i, b_i \in \mathbb{k}$. Firstly, we have

$$k_i = \sum_j m_j e_j^* (\phi_{11}(e_i)) + n_j e_j^* (\phi_{21}(e_i))$$

and

$$l_i = \sum_j m_j e_j^* (\phi_{12}(e_i)) + n_j e_j^* (\phi_{22}(e_i))$$

for any i . Further,

$$\sum_i e_j^* (\phi_{11}(e_i)) e_i^* = \phi_{11}^*(e_j^*)$$

by the definition of ϕ_{11}^* . Hence, we have

$$\begin{aligned} f &= \sum_j m_j \phi_{11}^*(e_j^*) y_1^* + n_j \phi_{21}^*(e_j^*) y_1^* \\ &\quad + \sum_j m_j \phi_{12}^*(e_j^*) y_2^* + n_j \phi_{22}^*(e_j^*) y_2^* \\ &\quad + \sum_i m_i y_1^* e_i^* + n_i y_2^* e_i^* \\ &= \sum_i m_i (y_1^* e_i^* + \phi_{11}^*(e_i^*) y_1^* + \phi_{12}^*(e_i^*) y_2^*) \\ &\quad + n_i (y_2^* e_i^* + \phi_{21}^*(e_i^*) y_1^* + \phi_{22}^*(e_i^*) y_2^*), \end{aligned}$$

which completes the proof. \square

Proposition 2.5. *Suppose that A is a Koszul algebra and $B = A_P[y_1, y_2; \sigma]$ is a trimmed right double Ore extension of A . Then,*

- (1) the map $A^! \rightarrow B^!$ is injective;
- (2) $B^! = A^! \oplus A^! y_1^* \oplus A^! y_2^* \oplus A^! y_1^* y_2^* = A^! \oplus y_1^* A^! \oplus y_2^* A^! \oplus y_1^* y_2^* A^!$. Moreover, $B^!$ is a free right (and left) $A^!$ -module with a basis $\{1, y_1^*, y_2^*, y_1^* y_2^*\}$.

Proof. As a matter of fact, the proof is similar to [LSV96, Proposition 2.5]. So we just give a sketch of it. First of all, since B is a free left A -module with basis $\{y_1^i y_2^j; i, j \geq 0\}$ by definition, the Hilbert series of B is equal to the Hilbert series of $A \otimes \mathbb{k}[y_1, y_2]$, i.e.,

$$H_B(t) = \frac{H_A(t)}{(1-t)^2}.$$

It is well known that there is a functional equation on Hilbert series

$$H_C(t) H_{C^!}(-t) = 1$$

for a Koszul algebra C . Since both A and B are Koszul algebras, so we have $H_{B^!}(t) = (1+t)^2 H_{A^!}(t)$. Thus, statement (1) follows from the following equivalent conditions whose proofs are similar to those of [LSV96, Theorem 2.6]:

- (i) $H_{B^!}(t) = (1+t)^2 H_{A^!}(t)$;
- (ii) $A^! \rightarrow B^!$ is injective;
- (iii) $A_3^! \rightarrow B_3^!$ is injective.

Moreover, there is a surjective algebra homomorphism $A^! \coprod \mathbb{k}\langle Y_1, Y_2 \rangle \rightarrow B^!$ from the coproduct of $A^!$ and the free algebra $\mathbb{k}\langle Y_1, Y_2 \rangle$ to $B^!$, which sends Y_i to y_i^* for $i = 1, 2$. By Lemma 2.4, the kernel of this map is the ideal generated by

$$\{Y_1^2 + qY_2Y_1, Y_1Y_2 + pY_2Y_1, Y_2^2\} \cup \{Y_j e_i^* + \phi_{j1}^*(e_i^*)Y_1 + \phi_{j2}^*(e_i^*)Y_2; j = 1, 2, i = 1, \dots, n\}.$$

Now consider the algebra

$$E := (A^! \coprod \mathbb{k}\langle Y_1, Y_2 \rangle) / (Y_j e_i^* + \phi_{j1}^*(e_i^*)Y_1 + \phi_{j2}^*(e_i^*)Y_2; j = 1, 2, i = 1, \dots, n).$$

It is not hard to see that E is a free right (also left) $A^!$ -module with the same basis as the free algebra $\mathbb{k}\langle Y_1, Y_2 \rangle$. Then, statement (2) will follow from the claim: the following three elements

$$Y_1^2 + qY_2Y_1, \quad Y_1Y_2 + pY_2Y_1, \quad Y_2^2$$

are normal elements in the algebra E . To see this, it suffices to note that ϕ_{ij} 's, determined by Definition 1.6(2).(iv), satisfy the relations dual to $R3.1$ - $R3.3$ in [ZZ08, p. 2674]. \square

In order to state and prove the main result of this section, we need more notation. Assume that $(\phi_{ij}^{lk})_{n \times n}$ is the matrix of the restriction of the k -linear map ϕ_{ij} to V , i. e.,

$$\phi_{ij}(e_l) = \sum_k \phi_{ij}^{lk} e_k$$

for each l . Both σ^* and ϕ^* are algebra homomorphisms from $A^!$ to $M_{2 \times 2}(A^!)$. Since δ is a base element of the highest nonzero component $A_d^!$ of $A^!$, we assume:

$$(2.7) \quad \sigma^*(\delta) = \begin{pmatrix} W\delta & X\delta \\ Y\delta & Z\delta \end{pmatrix}, \quad \phi^*(\delta) = \begin{pmatrix} W'\delta & X'\delta \\ Y'\delta & Z'\delta \end{pmatrix}$$

for some $W, X, Y, Z, W', X', Y', Z' \in \mathbb{k}$. Thus we have the following obvious property from Lemma 2.2.

Lemma 2.6. $\begin{pmatrix} W & X \\ Y & Z \end{pmatrix} \begin{pmatrix} W' & X' \\ Y' & Z' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$

In order to compute the Nakayama automorphism of $B^!$, we prove the following:

Lemma 2.7. (1) $\varepsilon := \delta y_1^* y_2^*$ is a basis element of the 1-dimensional space $B_{d+2}^!$.

(2) $\{\eta_1 y_1^* y_2^*, \eta_2 y_1^* y_2^*, \dots, \eta_m y_1^* y_2^*, \delta y_1^*, \delta y_2^*\}$ forms a basis of $B_{d+1}^!$.

(3) For any $1 \leq i, j \leq n$ and $m = 1, 2$, the following equations hold:

$$\left\{ \begin{array}{ll} e_i^* \eta_j y_1^* y_2^* \stackrel{(a)}{=} \delta_{ij} \varepsilon, & \eta_i y_1^* y_2^* e_j^* \stackrel{(b)}{=} \sum_{k,l} \left(\frac{q}{p} \phi_{21}^{kj} \phi_{11}^{lk} - \frac{1}{p} \phi_{21}^{kj} \phi_{12}^{lk} + \phi_{22}^{kj} \phi_{11}^{lk} \right) \lambda_{il} \varepsilon, \\ e_i^* \delta y_m^* \stackrel{(c)}{=} 0, & \delta y_m^* e_i^* \stackrel{(d)}{=} 0, \\ y_1^* \delta y_2^* \stackrel{(e_1)}{=} (-1)^d W' \varepsilon, & y_1^* \delta y_1^* \stackrel{(e_2)}{=} (-1)^d \left(\frac{q}{p} W' - \frac{1}{p} X' \right) \varepsilon, \\ y_2^* \delta y_2^* \stackrel{(e_3)}{=} (-1)^d Y' \varepsilon, & y_2^* \delta y_1^* \stackrel{(e_4)}{=} (-1)^d \left(\frac{q}{p} Y' - \frac{1}{p} Z' \right) \varepsilon, \\ \delta y_1^* y_1^* \stackrel{(f_1)}{=} \frac{q}{p} \varepsilon, & \delta y_1^* y_2^* \stackrel{(f_2)}{=} \varepsilon, \\ \delta y_2^* y_1^* \stackrel{(f_3)}{=} -\frac{1}{p} \varepsilon, & \delta y_2^* y_2^* \stackrel{(f_4)}{=} 0, \\ y_m^* \eta_j y_1^* y_2^* \stackrel{(g)}{=} 0, & \eta_j y_1^* y_2^* y_m^* \stackrel{(h)}{=} 0. \end{array} \right.$$

Proof. (1) is obvious by Proposition 2.5 (2). The equation (a) follows from the definition. Since $A^! \rightarrow B^!$ is injective, equations (c) and (d) hold naturally. Equations (g) and (h) follow from the relations (2) and (3) of Lemma 2.4. Now for (b), by relation (3) of Lemma 2.4 and Proposition 2.3, we have

$$\begin{aligned} y_1^* y_2^* e_j^* &= -y_1^* (\phi_{21}^*(e_j^*) y_1^* + \phi_{22}^*(e_j^*) y_2^*) \\ &= -\sum_k (\phi_{21}^{kj} y_1^* e_k^* y_1^* + \phi_{22}^{kj} y_1^* e_k^* y_2^*) \\ &= \sum_k (\phi_{21}^{kj} \phi_{11}^*(e_k^*) y_1^* y_1^* + \phi_{21}^{kj} \phi_{12}^*(e_k^*) y_2^* y_1^* + \phi_{22}^{kj} \phi_{11}^*(e_k^*) y_1^* y_2^*) \\ &= \sum_{k,l} (\phi_{21}^{kj} \phi_{11}^{lk} e_l^* y_1^* y_1^* + \phi_{21}^{kj} \phi_{12}^{lk} e_l^* y_2^* y_1^* + \phi_{22}^{kj} \phi_{11}^{lk} e_l^* y_1^* y_2^*) \\ &= \sum_{k,l} \left(\frac{q}{p} \phi_{21}^{kj} \phi_{11}^{lk} - \frac{1}{p} \phi_{21}^{kj} \phi_{12}^{lk} + \phi_{22}^{kj} \phi_{11}^{lk} \right) e_l^* y_1^* y_2^*. \end{aligned}$$

Thus, the equation (b) will follow from the assumption. Next we prove the rest equations. For a fixed j , suppose that $\eta_j = \sum_m \lambda_m e_{m_1}^* e_{m_2}^* \cdots e_{m_{d-1}}^*$ where $\lambda_m \in k$, then:

$$\begin{aligned} \begin{pmatrix} y_1^* \\ y_2^* \end{pmatrix} \delta &= \begin{pmatrix} y_1^* \\ y_2^* \end{pmatrix} e_j^* \eta_j \\ &= \sum_m \lambda_m \begin{pmatrix} y_1^* \\ y_2^* \end{pmatrix} e_j^* e_{m_1}^* e_{m_2}^* \cdots e_{m_{d-1}}^* \\ &= -\sum_m \lambda_m \phi^*(e_j^*) \begin{pmatrix} y_1^* \\ y_2^* \end{pmatrix} e_{m_1}^* e_{m_2}^* \cdots e_{m_{d-1}}^* \\ &= (-1)^2 \sum_m \lambda_m \phi^*(e_j^*) \phi^*(e_{m_1}^*) \begin{pmatrix} y_1^* \\ y_2^* \end{pmatrix} e_{m_2}^* \cdots e_{m_{d-1}}^* \\ &= \cdots \\ &= (-1)^d \sum_m \lambda_m \phi^*(e_j^*) \phi^*(e_{m_1}^*) \phi^*(e_{m_2}^*) \cdots \phi^*(e_{m_{d-1}}^*) \begin{pmatrix} y_1^* \\ y_2^* \end{pmatrix} \\ &= (-1)^d \sum_m \lambda_m \phi^*(e_j^* e_{m_1}^* e_{m_2}^* \cdots e_{m_{d-1}}^*) \begin{pmatrix} y_1^* \\ y_2^* \end{pmatrix} \end{aligned}$$

$$\begin{aligned}
&= (-1)^d \phi^* \left(e_j^* \sum_m \lambda_m e_{m_1}^* e_{m_2}^* \cdots e_{m_{d-1}}^* \right) \begin{pmatrix} y_1^* \\ y_2^* \end{pmatrix} \\
&= (-1)^d \phi^* (e_j^* \eta_j) \begin{pmatrix} y_1^* \\ y_2^* \end{pmatrix} \\
&= (-1)^d \phi^* (\delta) \begin{pmatrix} y_1^* \\ y_2^* \end{pmatrix}.
\end{aligned}$$

Hence, by the definition of ϕ^* , we obtain:

$$\begin{pmatrix} y_1^* \\ y_2^* \end{pmatrix} \delta = (-1)^d \begin{pmatrix} \phi_{11}^*(\delta) & \phi_{12}^*(\delta) \\ \phi_{21}^*(\delta) & \phi_{22}^*(\delta) \end{pmatrix} \begin{pmatrix} y_1^* \\ y_2^* \end{pmatrix} = (-1)^d \begin{pmatrix} W' \delta y_1^* + X' \delta y_2^* \\ Y' \delta y_1^* + Z' \delta y_2^* \end{pmatrix}.$$

Then, the other equations follow.

Finally, we show the statement (2). Suppose that:

$$a_1 \eta_1 y_1^* y_2^* + \cdots + a_n \eta_n y_1^* y_2^* + b_1 \delta y_1^* + b_2 \delta y_2^* = 0$$

for some coefficients, $a_1, \dots, a_n, b_1, b_2 \in \mathbb{k}$. By the construction of the Frobenius pair of $B^!$ (see (2.5)) and the equations (a), (c), $(e_1)-(e_4)$ and (g), we get $n+2$ linear equations with indeterminates $a_1, \dots, a_n, b_1, b_2$. From the fact that $\{\eta_1, \eta_2, \dots, \eta_n\}$ is a basis of $A_{d-1}^!$ and Lemma 2.6, it follows that this system of linear equations has only the zero solution. That is, the vectors $\eta_1 y_1^* y_2^*, \eta_2 y_1^* y_2^*, \dots, \eta_n y_1^* y_2^*, \delta y_1^*$ and δy_2^* are linear independent. On the other hand, since $H_{B^!}(t) = (1+t)^2 H_{A^!}(t)$ we have $\dim B_{d+1}^! = 2 \dim A_d^! + \dim A_{d-1}^! = n+2$. Hence, these vectors form a basis. \square

Proposition 2.8. *The restriction of the Nakayama automorphism $\mu_{B^!}$ to $A^!$ equals $\mu_{A^!}(\det_l \phi)^*$.*

Proof. It follows from the equations (a), (b), (c) and (d) in Lemma 2.7 and the nondegenerate bilinear form on $B^!$ (see (2.5)) that we have the following

$$\begin{cases} \langle e_i^*, \eta_j y_1^* y_2^* \rangle = \delta_{ij} a, & \langle \eta_i y_1^* y_2^*, e_j^* \rangle = \sum_{k,l} \left(\frac{q}{p} \phi_{21}^{kj} \phi_{11}^{lk} - \frac{1}{p} \phi_{21}^{kj} \phi_{12}^{lk} + \phi_{22}^{kj} \phi_{11}^{lk} \right) \lambda_{il}, \\ \langle e_i^*, \delta y_m^* \rangle = 0, & \langle \delta y_m^*, e_i^* \rangle = 0. \end{cases}$$

According to Lemma 2.7, $\{\eta_1 y_1^* y_2^*, \eta_2 y_1^* y_2^*, \dots, \eta_n y_1^* y_2^*, \delta y_1^*, \delta y_2^*\}$ is a basis of $B_{d+1}^!$, so we have

$$\mu_{B^!}(e_i^*) = \mu_{A^!}(\det_l \phi)^*(e_i^*)$$

for $i = 1, \dots, n$ by the definition of the Nakayama automorphism (see (1.1)) and that finishes the proof. \square

We need the following technical result although the proof is obvious.

Lemma 2.9. *Let $E = \mathbb{k} \oplus E_1 \oplus \cdots \oplus E_m$ be a graded Frobenius algebra. Suppose that $\{\alpha_1, \alpha_2\}$ and $\{\beta_1, \beta_2\}$ are bases of E_1 and E_{m-1} respectively. If*

$$\begin{cases} \langle \alpha_1, \beta_1 \rangle = a, & \langle \beta_1, \alpha_1 \rangle = e, \\ \langle \alpha_1, \beta_2 \rangle = b, & \langle \beta_2, \alpha_1 \rangle = f, \\ \langle \alpha_2, \beta_1 \rangle = c, & \langle \beta_1, \alpha_2 \rangle = g, \\ \langle \alpha_2, \beta_2 \rangle = d, & \langle \beta_2, \alpha_2 \rangle = h, \end{cases}$$

then the Nakayama automorphism of E is given by:

$$\begin{aligned}\mu(\alpha_1) &= \frac{de - cf}{ad - bc}\alpha_1 + \frac{af - be}{ad - bc}\alpha_2, \\ \mu(\alpha_2) &= \frac{dg - ch}{ad - bc}\alpha_1 + \frac{ah - bg}{ad - bc}\alpha_2.\end{aligned}$$

Proof. One just needs to observe that the Frobenius pair $\langle -, - \rangle$ is a nondegenerate bilinear form and therefore $ad - bc \neq 0$. \square

Proposition 2.10.

$$\begin{aligned}\mu_{B^!}(y_1^*) &= (-1)^{d+1} \left((qX + \frac{q}{p}X + \frac{1}{p}W)y_1^* + (qZ + \frac{q}{p}Z + \frac{1}{p}Y)y_2^* \right), \\ \mu_{B^!}(y_2^*) &= (-1)^{d+1} (pXy_1^* + pZy_2^*).\end{aligned}$$

Proof. Similar to the proof of Proposition 2.8, we want to compute the value of

$$\langle y_m^*, x \rangle, \quad \langle x, y_m^* \rangle$$

for $m = 1, 2$ and all $x \in B_{d+1}^!$. By equations (g) and (h) in Lemma 2.7,

$$\langle y_m^*, \eta_j y_1^* y_2^* \rangle = 0, \quad \langle \eta_j y_1^* y_2^*, y_m^* \rangle = 0$$

for $m = 1, 2$. That is to say, we arrive at the situation of Lemma 2.9. Then, by using equations $(e_1)-(e_4)$ and $(f_1)-(f_4)$ in Lemma 2.7 and by Lemma 2.9, we have

$$\begin{aligned}\mu_{B^!}(y_1^*) &= (-1)^d \left(\frac{qY' + \frac{q}{p}Y' - \frac{1}{p}Z'}{W'Z' - X'Y'} y_1^* + \frac{-qW' - \frac{q}{p}W' + \frac{1}{p}X'}{W'Z' - X'Y'} y_2^* \right), \\ \mu_{B^!}(y_2^*) &= (-1)^d \left(\frac{pY'}{W'Z' - X'Y'} y_1^* + \frac{-pW'}{W'Z' - X'Y'} y_2^* \right).\end{aligned}$$

Finally, the statement follows from the equation

$$\begin{pmatrix} W & X \\ Y & Z \end{pmatrix} = \frac{1}{W'Z' - X'Y'} \begin{pmatrix} W' & -X' \\ -Y' & Z' \end{pmatrix}$$

which is a sequence of Lemma 2.6. \square

Proposition 2.11. *The restriction of the Nakayama automorphism ν_B of B to A equals $(\det_r \sigma)^{-1} \nu$, and*

$$\begin{aligned}\nu_B(y_1) &= (qX + \frac{q}{p}X + \frac{1}{p}W)y_1 + pXy_2, \\ \nu_B(y_2) &= (qZ + \frac{q}{p}Z + \frac{1}{p}Y)y_1 + pZy_2.\end{aligned}$$

Proof. Due to Proposition 2.8 and Proposition 2.10, the restriction of Nakayama automorphism $\mu_{B^!}$ to $B_1^!$ has the form $\begin{pmatrix} P & 0 \\ 0 & Q \end{pmatrix}$, where $P \in M_d(\mathbb{k})$ and $Q \in M_2(\mathbb{k})$. Also, the same property holds for its dual by Equation 1.2. Then, the conclusion follows from Proposition 1.4 and Proposition 1.10. \square

Now we are ready to state the main result of this section.

Theorem 2.12. *Suppose that A is a Koszul AS-regular algebra with Nakayama automorphism ν . Let $B = A_P[y_1, y_2; \sigma]$ be a trimmed double Ore extension of A . Then B is Calabi-Yau if and only if the following two conditions are satisfied:*

- (1) $\det_r \sigma = \nu$;
- (2) $W = p$, $X = 0$, $Y = -(1 + \frac{1}{p})q$ and $Z = \frac{1}{p}$.

Proof. Since B is Koszul with finite global dimension, the Koszul bimodule complex provides a finite generated and finite length projective resolution of B as an B^e -module. That is, B is homologically smooth. Because B is connected graded, the inner automorphism must be the identity. Therefore, the statement is a sequence of Proposition 2.11. \square

Remark 2.13. *According to the assumption on constants W, X, Y and Z (see equation (2.7)) and the characterization of the homological determinant (see Proposition 1.5), the condition (2) of Theorem 2.12 is called a homological determinant type condition.*

Remark 2.14. *For a Koszul AS-regular algebra A with Nakayama automorphism ν , there exists a unique skew polynomial extension B such that B is Calabi-Yau by [GK13, GYZ14, HVZ13, LWW12, RRZ13]. Here, we consider the existence and the uniqueness of a Calabi-Yau trimmed double Ore extension of a Koszul AS-regular algebra.*

- (1) *Consider the trimmed double Ore extension $B = A_P[y_1, y_2; \sigma]$ with $P = (1, 0)$ and $\sigma = \begin{pmatrix} \nu & 0 \\ 0 & id \end{pmatrix}$. Then B is Calabi-Yau. But it is easy to see that B is an iterated Ore extension of A (see [ZZ09, Proposition 3.6] or its proof). Hence, we ask if there exists a nontrivial double Ore extension B (not an iterated one) such that B is Calabi-Yau? The answer is negative from the following example. Let $A = \mathbb{k}\langle x_1, x_2 \rangle / (x_2 x_1 - x_1 x_2 - x_1^2)$ be the Jordan plane. Then, there is only one nontrivial double Ore extension by the classification in [ZZ09], namely, the type $\mathbb{H} := A_P[y_1, y_2; \sigma]$ with $P = (-1, 0)$ and σ given by the matrix $\begin{pmatrix} 0 & 0 & h & 0 \\ 0 & 0 & hf & h \\ h & 0 & 0 & 0 \\ hf & h & 0 & 0 \end{pmatrix}$ with $0 \neq h \in \mathbb{k}$ and $f \in \mathbb{k}$. Moreover, $\det_r(\sigma) = \nu$ if and only if $h^2 = 1$ and $f = 1$. Now,*

$$\sigma^*(\delta) = \begin{pmatrix} -\delta & 0 \\ 0 & -\delta \end{pmatrix}.$$

Therefore, there is no Calabi-Yau algebra of the type \mathbb{H} by Theorem 2.12.

- (2) *For the uniqueness, let $A = \mathbb{k}\langle x_1, x_2 \rangle / (x_2 x_1 + x_1 x_2)$ be the quantum plane and $B := A_P[y_1, y_2; \sigma]$ with $P = (-1, 0)$, where σ is given by the matrix $\begin{pmatrix} 0 & -g & 0 & f \\ g & 0 & f & 0 \\ 0 & f & 0 & -g \\ f & 0 & g & 0 \end{pmatrix}$ with $f, g \in \mathbb{k}$ and $f^2 \neq g^2$. So B is of type \mathbb{N} in the classification of [ZZ09]. In this case, $\det_r(\sigma) = \nu$ if and only if $f^2 - g^2 = -1$.*

If such a condition is satisfied, then $\sigma^*(\delta) = \begin{pmatrix} -\delta & 0 \\ 0 & -\delta \end{pmatrix}$. Hence, B is Calabi-Yau if and only if $f^2 - g^2 = -1$ by Theorem 2.12. Therefore, the double Ore extension, which is Calabi-Yau, of a Koszul AS-regular algebra may not be unique if it exists.

Remark 2.15. By the example in Remark 2.14 (1), we can see the condition (1) and condition (2) in Theorem 2.12 are independent.

To end this section, we return to discuss the Nakayama automorphism and the Calabi-Yau property of the skew polynomial extension. For a twist Calabi-Yau algebra A with Nakayama automorphism ν , it was proved in [LWW12, Theorem 3.3] that the Nakayama automorphism of the Ore extension $D = A[t; \theta, \delta]$ is given by

$$\nu_D(x) = \begin{cases} \theta^{-1} \circ \nu(x), & x \in A; \\ ax + b, & x = t, \end{cases}$$

for some $a, b \in A$ with a invertible. It was also remarked there that if $\delta = 0$, then $\nu_D(t) = at$. Now if we restrict to Koszul algebras, we can describe the Nakayama automorphism more explicitly as follows.

Proposition 2.16. Suppose that A is a Koszul AS-regular algebra with Nakayama automorphism ν , θ is a graded algebra automorphism of A and $D = A[t; \theta]$. The Nakayama automorphism ν_D of D is given by:

$$\nu_D(x) = \begin{cases} \theta^{-1} \circ \nu(x), & x \in A \\ (\text{hdet } \theta)x, & x = t. \end{cases}$$

Proof. We only give a sketch of proof since it is similar to the one of Proposition 2.11. Suppose that $D = T(V \oplus \mathbb{k}t) / \langle R_D \rangle$. The generating relations in D are of two types: $te_i - \theta(e_i)t$ ($1 \leq i \leq n$) and the relations for A . Obviously, $\{e_1^*, e_2^*, \dots, e_n^*, t^*\}$ forms a basis for $D_1^!$. By [LSV96, Proposition 2.4], the defining relations for $D^!$ consist of

- (1) the relations for $A^!$;
- (2) $\{t^*e_i^* + (\theta^{-1})^*(e_i^*)t^* \mid 1 \leq i \leq n\}$;
- (3) $\{(t^*)^2\}$.

By [LSV96, Proposition 2.5], $D^!$ is a free $A^!$ -module with basis $\{1, t^*\}$. Hence, δt^* is a base element of the 1-dimensional space $D_{d+1}^!$, denoted it by ε . Assume that $(b_{ij})_{n \times n}$ is the matrix of the restriction of θ^{-1} to V , i. e.,

$$(2.8) \quad \theta^{-1}(e_i) = \sum_j b_{ij} e_j$$

for each i . Then, we have

- (1) $\{\eta_1 t^*, \eta_2 t^*, \dots, \eta_n t^*, \delta\}$ is a basis of $D_d^!$;

(2) the following equations hold:

$$\begin{cases} e_i^* \eta_j t^* = \delta_{ij} \varepsilon, & \eta_i t^* e_j^* = -\sum_k b_{kj} \lambda_{ik} \varepsilon, \\ e_i^* \delta = 0, & \delta e_i^* = 0, \\ t^* \delta = (-1)^d (\text{hdet}(\theta))^{-1} \varepsilon, & \delta t^* = \varepsilon, \\ t^* \eta_j t^* = 0, & \eta_j t^* t^* = 0. \end{cases}$$

Using the same way as in the proof of Proposition 2.8 and Proposition 2.10, the Nakayama automorphism $\mu_{D^!}$ of $D^!$ is given by:

$$\mu_{D^!}(\alpha) = \begin{cases} -\mu_{A^!} \circ (\theta^{-1})^*(\alpha), & \alpha \in A^! \\ (-1)^d (\text{hdet } \theta) \alpha, & \alpha = t^*. \end{cases}$$

The last step is to transfer $\mu_{D^!}$ to the Nakayama automorphism ν_D of D by Proposition 1.4. \square

Note that the homological determinant of the Nakayama automorphism of a Koszul algebra is equal to 1 [RRZ13, Theorem 0.4]. Thus, we arrive at the following result which was proved in [GK13, GYZ14, HVZ13, LWW12, RRZ13]:

Theorem 2.17. *Suppose that A is a Koszul AS-regular algebra with Nakayama automorphism ν , and θ is a graded algebra automorphism of A and $D = A[t; \theta]$. Then, D is Calabi-Yau if and only if $\theta = \nu$.*

3. SKEW LAURENT EXTENSIONS

In this section, we consider the Calabi-Yau property of both the Ore localization of $A[t; \theta]$ and the algebra $A_P[y_1^{\pm 1}, y_2^{\pm 1}; \sigma]$ with σ a diagonal type. For a skew polynomial extension $A[t; \theta]$ of an algebra A , the multiplicatively closed set $\{t^i; i \in \mathbb{N}\}$ is an Ore set. The localization of $A[t; \theta]$ with respect to this Ore set is just the skew Laurent polynomial extension $A[t^{\pm 1}; \theta]$. Farinati proposed a general notion of a noncommutative localization in [F05]. It was proved there that the Van den Bergh duality is preserved under such a localization and the corresponding dualizing module is also explicitly described. The Ore localization is just an example of a noncommutative localization [F05, Example 8].

Proposition 3.1. *Suppose A is a Koszul AS-regular algebra of dimension d and $D = A[t; \theta]$ is a skew polynomial extension of A . The Nakayama automorphism $\tilde{\nu}$ of $A[t^{\pm 1}; \theta]$ is given by*

$$\tilde{\nu}(x) = \begin{cases} \nu_D(x), & x \in D \\ \frac{1}{\text{hdet } \theta} x, & x = t^{-1}. \end{cases}$$

Proof. By assumption and [F05, Theorem 6], we have:

$$\text{Ext}_{E^e}^i(E, E^e) \cong \begin{cases} 0, & i \neq d+1 \\ E \otimes_D D^\nu \otimes_D E(d+1), & i = d+1, \end{cases}$$

where E stands for the algebra $A[t^{\pm 1}; \theta]$. Thus, the claim follows from the description of the Nakayama automorphism of D in Proposition 2.16. \square

Theorem 3.2. *Suppose that A is a Koszul AS-regular algebra with Nakayama automorphism ν , θ is a graded algebra automorphism of A . Then, $A[t^{\pm 1}; \theta]$ is graded Calabi-Yau if and only if there exists an integer n such that $\theta^n = \nu$ and the homological determinant of θ equals 1.*

Proof. It follows from the proof of [F05, Theorem 6] that $A[t^{\pm 1}; \theta]$ is homologically smooth. Then the proof mainly relies on the description of the Nakayama automorphisms of algebras $A[t; \theta]$ and $A[t^{\pm 1}; \theta]$ proved in Proposition 2.16 and Proposition 3.1 respectively. First of all, it is easy to see that the only invertible elements in $\mathbb{k}[t^{\pm 1}]$ are monomials. Suppose that $A[t^{\pm 1}; \theta]$ is Calabi-Yau. Then, its' Nakayama automorphism $\tilde{\nu}$ is inner, i.e., there exists an integer $n \in \mathbb{Z}$ such that $\tilde{\nu}(x) = t^n x t^{-n}$ for each $x \in A[t^{\pm 1}; \theta]$. In particular, $\tilde{\nu}(t) = t$. Therefore, $\text{hdet}(\theta) = 1$ by Proposition 2.16. If n is nonnegative, then for each $x \in A$ we have

$$\begin{aligned} \tilde{\nu}(x) &= \theta^{-1}\nu(x) = t^n x t^{-n} \\ &= t^n (t^{-1}\theta(x)t) t^{-n} \\ &= t^{n-1}\theta(x)t^{1-n} \\ &= \cdots = \theta^n(x). \end{aligned}$$

Hence, $\nu(x) = \theta^{n+1}(x)$. Similarly, the claim also holds for the case when n is a negative integer.

Conversely, if $\theta^n = \nu$ for some integer n and the homological determinant of σ equals 1, then $\tilde{\nu}(t) = t$. Next, for each $x \in A$, we have

$$\tilde{\nu}(x) = \theta^{-1}\nu(x) = \theta^{n-1}(x).$$

But in $A[t^{\pm 1}; \theta]$, $\theta(x) = txt^{-1}$. That is, both θ and its inverse are inner. Therefore, $\tilde{\nu}$ is an inner automorphism. The proof is completed. \square

Example 3.3. *Let $A = \mathbb{k}\langle x, y \rangle / (yx - xy - x^2)$ be the Jordan plane. It is a twisted Calabi-Yau algebra of dimension 2 whose Nakayama automorphism ν given by $\nu(x) = x$ and $\nu(y) = 2x + y$. Then, $A[t; \theta]$ is Calabi-Yau if and only if $\theta = \nu$ by Theorem 2.17. It is not hard to see that each graded automorphism θ of A has the form $\theta(x) = ax$ and $\theta(y) = bx + ay$ for some $a, b \in \mathbb{k}$. By Proposition 1.5, the homological determinant of θ equals a^2 . Thus, $A[t^{\pm 1}; \theta]$ is Calabi-Yau if and only if θ is either given by*

$$\begin{cases} \theta(x) = x \\ \theta(y) = \pm \frac{2}{n}x + y \end{cases}$$

for some nonzero integer n , or given by

$$\begin{cases} \theta(x) = -x \\ \theta(y) = \pm \frac{2}{n}x - y \end{cases}$$

for some even integer n .

Finally, we are going to study the localization or the quotient ring of the double Ore extension B with respect to the Ore set generated by new generators. However, we can only do this in some special case as follows.

Proposition 3.4. *Let $B = A_P[y_1, y_2; \sigma]$ be a trimmed double Ore extension with $P = (p, 0)$ and $\sigma = \begin{pmatrix} \tau & 0 \\ 0 & \xi \end{pmatrix}$. Then,*

- (1) *Both τ and ξ are automorphisms of A . Moreover, they commute with each other.*
- (2) *The multiplicatively closed set $S := \{ay_1^{n_1}y_2^{n_2}; a \in k, n_1, n_2 \in \mathbb{Z}_{\geq 0}\}$ is an Ore set.*
- (3) *The quotient ring B_S of B with respect to S exists.*

Proof. Since B is a trimmed double Ore extension of A , σ is invertible according to Lemma 1.9. Hence, both τ and ξ are automorphisms of A . By the definition of right determinant of σ (see (1.5)) and its equivalent description in Proposition 1.10, we have $\tau\xi = \xi\tau$. The rest is easy. \square

In fact, the algebra $B = A_P[y_1, y_2; \sigma]$ considered above is an iterated skew polynomial extension $A[y_1; \tau][y_2; \xi']$ where ξ' is the automorphism of $A[y_1; \tau]$ defined as follows

$$\xi'(x) = \begin{cases} \xi(x), & x \in A; \\ px, & x = y_1. \end{cases}$$

If $p = 1$, then the quotient ring B_S is isomorphic to the iterated skew Laurent ring $A[y_1^{\pm 1}, y_2^{\pm 1}; \tau, \xi]$ (see [GW04, p.23-24]). For the case of $p \neq 1$, we can also construct the iterated skew Laurent ring and denote it $A_P[y_1^{\pm 1}, y_2^{\pm 1}; \tau, \xi]$ or just $A_P[y_1^{\pm 1}, y_2^{\pm 1}; \sigma]$. Similarly, the quotient ring B_S in the above Proposition is isomorphic to the iterated skew Laurent ring $A_P[y_1^{\pm 1}, y_2^{\pm 1}; \sigma]$.

Theorem 3.5. *Suppose that A is a Koszul AS-regular algebra with Nakayama automorphism ν , $B = A_P[y_1, y_2; \sigma]$ is a trimmed double Ore extension with $P = (p, 0)$ and $\sigma = \begin{pmatrix} \tau & 0 \\ 0 & \xi \end{pmatrix}$ and $B_S = A_P[y_1^{\pm 1}, y_2^{\pm 1}; \sigma]$. Then, B_S is Calabi-Yau if and only if there exist two integers m, n such that the following conditions are satisfied:*

- (1) $\tau^n \xi^m = \nu$;
- (2) $\text{hdet}(\tau) = p^m$ and $\text{hdet}(\xi) = 1/p^n$.

Proof. Observe that for the given automorphism $\sigma : A \rightarrow M_{2 \times 2}(A)$, the induced algebra homomorphism σ^* from $A^!$ to $M_{2 \times 2}(A^!)$ has the form $\begin{pmatrix} \tau^* & 0 \\ 0 & \xi^* \end{pmatrix}$, where τ^* and ξ^* are automorphisms of $A^!$ induced by τ and ξ respectively. By Equation (2.7) and Proposition 1.5, we have

$$W = \text{hdet } \tau, \quad X = 0, \quad Y = 0, \quad \text{and} \quad Z = \text{hdet } \xi.$$

It follows from the assumption and Proposition 2.11 that the Nakayama automorphism of B is given as follows:

$$\nu_B(x) = \begin{cases} (\tau\xi)^{-1} \circ \nu(x), & x \in A; \\ \frac{1}{p}(\text{hdet } \tau)x, & x = y_1; \\ p(\text{hdet } \xi)x, & x = y_2. \end{cases}$$

Then, by using [F05, Theorem 6] again, the Nakayama automorphism $\tilde{\nu}$ of B_S is given by

$$\tilde{\nu}(x) = \begin{cases} \nu_B(x), & x \in B \\ \frac{p}{\text{hdet } \tau} x, & x = y_1^{-1} \\ \frac{1}{p \text{hdet } \xi} x, & x = y_2^{-1} \end{cases}$$

Note that the only invertible elements in B_S are monomials $a_{nm} y_1^n y_2^m$ for some $a_{nm} \in \mathbb{k}$ and $n, m \in \mathbb{Z}$.

Suppose that B_S is Calabi-Yau. Then, its' Nakayama automorphism $\tilde{\nu}$ is inner, i.e., there exists integer $m, n \in \mathbb{Z}$ such that $\tilde{\nu}(x) = y_1^n y_2^m x y_2^{-m} y_1^{-n}$ for each B_S . In particular, $\tilde{\nu}(y_1) = y_1^n y_2^m y_1 y_2^{-m} y_1^{-n} = \frac{1}{p}(\text{hdet } \tau) y_1$. Then, $\text{hdet}(\tau) = p^{m+1}$ since now the relation between y_1 and y_2 is $y_2 y_1 = p y_1 y_2$. Similarly, we can prove $\text{hdet}(\xi) = 1/p^{n+1}$. Now, without loss of generality we suppose that both n and m are nonnegative. Then, for each element $x \in A$ we have

$$\begin{aligned} (\tau \xi)^{-1} \circ \nu(x) &= \tilde{\nu}(x) \\ &= y_1^n y_2^m x y_2^{-m} y_1^{-n} \\ &= y_1^n y_2^{m-1} \xi(x) y_2^{1-m} y_1^{-n} \\ &= \dots \\ &= y_1^n \xi^m(x) y_1^{-n} \\ &= \dots \\ &= \tau^n \xi^m(x). \end{aligned}$$

Hence, $\nu = \tau^{n+1} \xi^{m+1}$.

The proof for converse direction is similar. \square

In general, one can construct an iterated skew polynomial extension as follows. Suppose that $\theta_1, \dots, \theta_m$ are commutative graded automorphism of A . First of all, let $R_1 = A[y_1; \theta_1]$. Then, extend θ_2 to an algebra automorphism θ'_2 of R_1 such that $\theta'_2|_A = \theta_2$ and $\theta'_2(y_1) = y_1$. Now let $R_2 = A[y_1; \theta_1][y_2; \theta'_2]$. In this way, one can construct R_i for $i = 1, 2, \dots, m$, such that, for $i < m$, $R_{i+1} = R_i[y_{i+1}; \theta'_{i+1}]$, where θ'_{i+1} is the automorphism of R_i satisfying $\theta'_{i+1}|_A = \theta_{i+1}$ and $\theta'_{i+1}(y_j) = y_j$ for $j = 1, \dots, i$. In this way, we obtain:

$$R = R_m = A[y_1; \theta_1][y_2; \theta'_2] \cdots [y_m; \theta'_m].$$

In order to describe the basic data that determine A , one writes R in a different way as follows:

$$R = A[y_1, \dots, y_m; \theta_1, \dots, \theta_m].$$

Note that $y_i y_j = y_j y_i$ for all i, j , and that $y_i a = \theta_i(a) y_i$ for all i and all $a \in A$. The quotient ring R_S of R with respect to the multiplicatively closed set $S := \{y_1^{n_1} \cdots y_m^{n_m}; n_1, \dots, n_m \in \mathbb{Z}_{\geq 0}\}$ exists and is isomorphic to the iterated skew Laurent ring $A[y_1^{\pm 1}, \dots, y_m^{\pm 1}; \theta_1, \dots, \theta_m]$. For more details, we refer to [GW04, p.23-24]. In the following, we will give a criterion for such an iterated skew polynomial extension of a Koszul AS-regular algebra to be Calabi-Yau.

Theorem 3.6. *Suppose that A is a Koszul AS-regular algebra with Nakayama automorphism ν , $R = A[y_1, \dots, y_m; \theta_1, \dots, \theta_m]$ and $S := \{y_1^{n_1} \cdots y_m^{n_m}; n_1, \dots, n_m \in \mathbb{Z}_{\geq 0}\}$. Then,*

(1) *the Nakayama automorphism ν_R of R is given by*

$$\nu_R(x) = \begin{cases} (\theta_1 \circ \cdots \circ \theta_m)^{-1} \circ \nu(x), & x \in A \\ (\text{hdet } \theta_i) y_i, & x = y_i, 1 \leq i \leq m; \end{cases}$$

(2) *R is Calabi-Yau if and only if $\theta_1 \circ \cdots \circ \theta_m = \nu$ and $\text{hdet } \theta_i = 1$ for all i ;*

(3) *R_S is Calabi-Yau if and only if*

(i) *$\text{hdet}(\theta_i) = 1$ for all i , and*

(ii) *there exists a series of integers k_1, \dots, k_m such that $\theta_1^{k_1} \cdots \theta_m^{k_m} = \nu$.*

Proof. By Proposition 2.16, the Nakayama automorphism ν_{R_2} of R_2 is given by:

$$\nu_{R_2}(x) = \begin{cases} (\theta'_2)^{-1} \circ \nu_{R_1}(x), & x \in R_1 \\ (\text{hdet } \theta'_2)x, & x = y_2. \end{cases}$$

It follows from the construction of θ'_2 and the description of the Nakayama automorphism ν_{R_1} of R_1 that

$$\nu_{R_2}(x) = \begin{cases} (\theta_2 \theta_1)^{-1} \circ \nu(x), & x \in A; \\ \text{hdet } \theta_1 x, & x = y_1; \\ (\text{hdet } \theta'_2)x, & x = y_2. \end{cases}$$

On the other hand, according to the proof of Theorem 3.5, the action of Nakayama automorphism ν_{R_2} of R_2 on y_2 is $(\text{hdet } \theta_2)y_2$. Hence, $\text{hdet } \theta'_2 = \text{hdet } \theta_2$. Repeating this process, we obtain statement (1). Statement (2) follows from Part (1). The proof of Part (3) is similar to the proof of Theorem 3.5. \square

Note that a typical example of such a Calabi-Yau localization R_S is the smash product of a Koszul AS-regular algebra with a free abelian group algebra. For example, those Hopf algebras in the classification of Calabi-Yau pointed Hopf algebras of finite Cartan type in [YZ11].

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